II. *The Problem of Finite Focal Depth revealed by Seismometers.*

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THE results discussed in this paper were obtained about five years ago. Except for a brief reference in 'B.A. Reports,' 1917 ("Seismology") they have not been published. There were two reasons for this delay: (1) additional data were desired in a matter of somewhat critical importance in the measured properties of the earth; (2) my official duties left little spare time for the pursuit of a purely scientific branch of seismology which required a good deal of tentative numerical computation.

At the present date it appears that one must abandon all hope that additional results which were expected from Russian observatories can be obtained. Moreover, it appears very doubtful if relevant data from any of the Allied countries can be expected for some years to come. Accordingly, publication of the results so far obtained now seems desirable, and may serve to show how urgent is the need for the equipment of a few seismological observatories capable of obtaining the data that are wanted.

A brief introduction to the problem is necessary, although it covers ground which is fairly familiar to those interested in seismometry.

If we have complete data giving the brachistochronic time for a seismic ray (say the P wave) to travel from a given point on the earth's surface to any other point on the surface, it is possible to calculate the way in which the speed of propagation of the ray varies with the depth. Two methods are open: (1) we may use the differential equation for the path of the ray, or (2) we may use the integral equation obtained from this. Both methods correctly carried out must give the same result, and it is merely a question of convenience which one adopts. By using a comparatively rough graphical method based on (1), WIECHERT and ZÖPPRITZ showed, about 14 years ago, from their accumulated data that the speed for the P wave increases from 7·17 km./sec. at the surface to 12·7 km./sec. at a depth of 1500 km., while from 1500 km. to over 3000 km. depth the speed increases but slightly. No data are available for investigating greater depths.

More recently KNOTT ('Roy. Soc. Proc.,' 1918–1919) applied the second method to the same data as was used by ZÖPPRITZ, and his results do not differ materially from those of the earlier and rougher method.

A very important supposition has to be made, however, before either of these methods can be applied. That is, that the true focus is either at the surface or so near the surface that a small correction can be made for it. If, however, the focus is at a considerable

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and unknown depth both methods fail, as is obvious from the consideration that, since all the rays we can observe must have passed out from the focus, we have no data for the comparatively large range of distance AB on the surface for which brachistochronic rays would *not* penetrate as deep as does the observed seismic ray. In fact, another unknown element enters into the connexion between the observed time curve and the variation of speed with depth.

A finite depth of focus implies a minimum angle of emergence at some point on the earth's surface and a point of inflexion on the time curve, and *vice versa*. Now, in considering ZÖPPRITZ's accepted time curve for P, we recognize that up to $\Delta = 1000$ km. the curve is probably hypothetical, but from 1000 km. to 13,000 km. there is no indication of a point of inflexion. It is not until we come to consider GALITZIN's direct measurements of the angle of emergence that we are confronted with a most marked minimum angle of emergence near $\Delta = 4000$ km., implying a point of inflexion on the time curve and a very considerable depth of focus. As GALITZIN's observations form the whole basis of this paper, they are reproduced here (although published elsewhere), as the reader may desire to have them convenient for direct reference.

TABLE I.

Epicentral distance. Δ in kilometres.	For P.		
	e from time curve.	$\bar{e}$ computed.	$\bar{e}$ observed at Pulkovo.
	°	°	°
0	0	22	—
500	11	23	—
1,000	21	27	—
1,500	30	32	—
2,000	37	37	—
2,500	44	42	48
3,000	49	47	44
3,500	53	52	43
4,000	57	54	42
4,500	60	58	43
5,000	63	60	44
5,500	65	62	46
6,000	65	62	48
6,500	65	63	51
7,000	65	63	54
7,500	66	63	58
8,000	66	64	62
8,500	67	64	65
9,000	67	65	67
9,500	68	66	68
10,000	69	67	70
10,500	70	67	71
11,000	70	68	72
11,500	71	69	72
12,000	72	70	73
12,500	73	71	73
13,000	74	72	74

The quantity $\bar{e}$ is called the apparent angle of emergence at the surface, and is defined by

$$\tan \bar{e} = Z/H$$

where

Z is the observed vertical component of displacement

and

H is the observed horizontal component of displacement.

Thus $\bar{e}$ is the direct subject of measurement by vertical and horizontal seismometers.

The angle e is called the true angle of emergence of the brachistochronic ray. It cannot be directly measured, but may be calculated from the time curve for P by the formula

$$\cos e = V_1 \frac{dT}{d\Delta},$$

where V_1 is the speed for longitudinal waves at the surface, and T the time is supposed expressed in terms of the epicentral distance Δ.

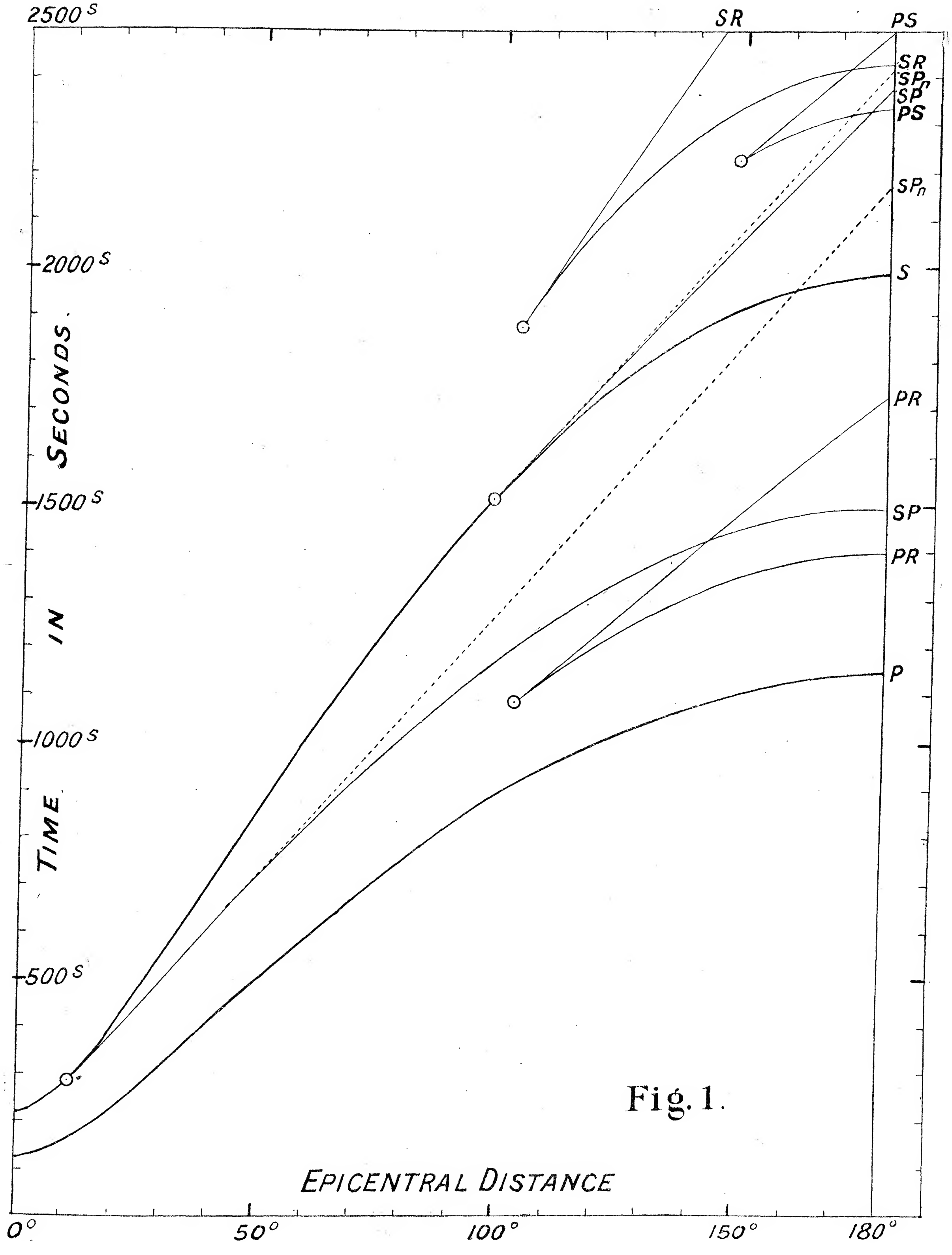
When the conditions of reflexion are examined it can be shown that for a longitudinal ray incident

$$\cos e = \frac{V_1}{V_2} \{\tfrac{1}{2}(1 - \sin \bar{e})\}^{\frac{1}{2}},$$

where V_2 is the surface speed of transversal waves.

It is seen from the table that the values of $\bar{e}$ calculated from ZÖPPRITZ's curve do not agree with the values of $\bar{e}$ directly measured at Pulkovo. The discrepancy is so marked that we may set aside the supposition that the Pulkovo values are merely instrumental errors. In a matter so important GALITZIN would hardly have published them if he had not felt assured that they were substantially correct. There remain two alternatives: (1) that the ratio V_1/V_2 for Pulkovo depends on the angle of impingence in such a way as to exactly annul the discrepancy. The probability of such compensation of actual facts to explain a theoretical formula must be regarded as small, and so we are left with alternative (2) that within the limits of possible error in the time curve we can modify it so as to agree with the direct measures of $\bar{e}$. We shall show that this alternative is quite possible within quite a large range of Δ. But we must at once point out the somewhat startling consequence of accepting the Pulkovo numbers as correct.

It has been shown that a ray which emerges with a minimum angle must have set out from the focus in a direction at right angles to the radius vector from the earth's centre to the focus. Thus for a minimum angle at $\Delta = 4000$ km. we find that even for a uniform earth the depth of focus required is about 0·2 of the earth's radius, or about 1250 km. The actual value may be a little less or a little more, according to the way in which speed varies with the depth. Anyhow, this is a much larger estimate of depth than has formerly been suggested, viz., of order less than 100 km.



Uniform earth R = 6370 km. Focal depth = 0·2 R. $V_1/V_2 = \sqrt{3}$. V_1 = 10 km. per sec.

A number of novel consequences with regard to reflexion follow if we admit such a great depth of focus, so that before showing how a time curve can be deduced from the Pulkovo observed angles $\bar{e}$, we may within advantage consider what is to be expected in a uniform earth, as thereby we shall be in a better position to deal with what may be inferred from the observed data.

We shall select for discussion a uniform earth $R = 6370$ km., depth of focus $= 0 \cdot 2$ R, with V_1/V_2 having the theoretical ratio $\sqrt{3}$, while V_1 is taken as 10 km./sec. These numbers are taken partly for convenience of calculation and partly to get as near as possible to the actual case.

The times for P and S may be computed for different epicentral distances from the trigonometrical formula for the paths traversed. The results for epicentral distances from 0 degree up to 180 degrees are shown in fig. 1. We may note that the point of inflexion on the time curve is very ill-defined, and might easily escape detection by direct observations of the time.

The direct measurement of the angle of emergence is, however, fairly precise, and from such measurements we can, in fact, calculate the time curve more accurately than we can determine it by direct observations of the time.

We may further note that the time increases but slowly for the first 1000 km., and since the angle of impingence for this region is not far short of 90 degrees, the true P might escape observation by horizontal seismographs, since the ground motion is almost entirely vertical.

Passing to waves reflected at the surface, we consider first waves which maintain their longitudinal character throughout. We may call them PR . . . PR_n waves. The simplest way of computing is to choose the point at which the first reflexion takes place and then calculate the epicentral distance to the final point of emergence (the station). The results are shown in figs. 2 and 3, where, in order to lead up to the large depth of focus, we have first shown the effect for depth $0 \cdot 01$ R (about 64 km.). We find that for this depth we cannot get a reflexion at all until Δ is about 23 degrees, and that for $\Delta > 23$ degrees there are two PR waves, the reflexion taking place at two different points, and they occur at different times. For depth as small as $0 \cdot 01$ R we see that we can proceed to PR_n, where n is moderately large.

When we pass to a focal depth $0 \cdot 2$ R we find that the smallest epicentral distance for which we can get PR is 103 degrees, and beyond this we have two PR's. But when we try to calculate the PR_2 we find that the least epicentral distance is over 180 degrees, and so we stop. The corresponding times for PR_1 are calculated and shown in fig. 1, and we note that the earlier arrival refers to the PR_1 which is reflected at the smaller distance from the epicentre. Figs. 2 and 3 are equally applicable to S waves in which the vibration is at right angles to the diametral plane through focus and station. The times for the SR waves are shown in fig. 1.

We consider next waves which undergo change from longitudinal to transversal, or *vice versa*, on reflexion. PS or SP, which for a very shallow focus would arrive together

on the seismogram, are now totally separate phenomena for a deep focus. Fig. 5 shows that PS cannot occur until Δ is 149 degrees, and beyond this there are two PS's for any

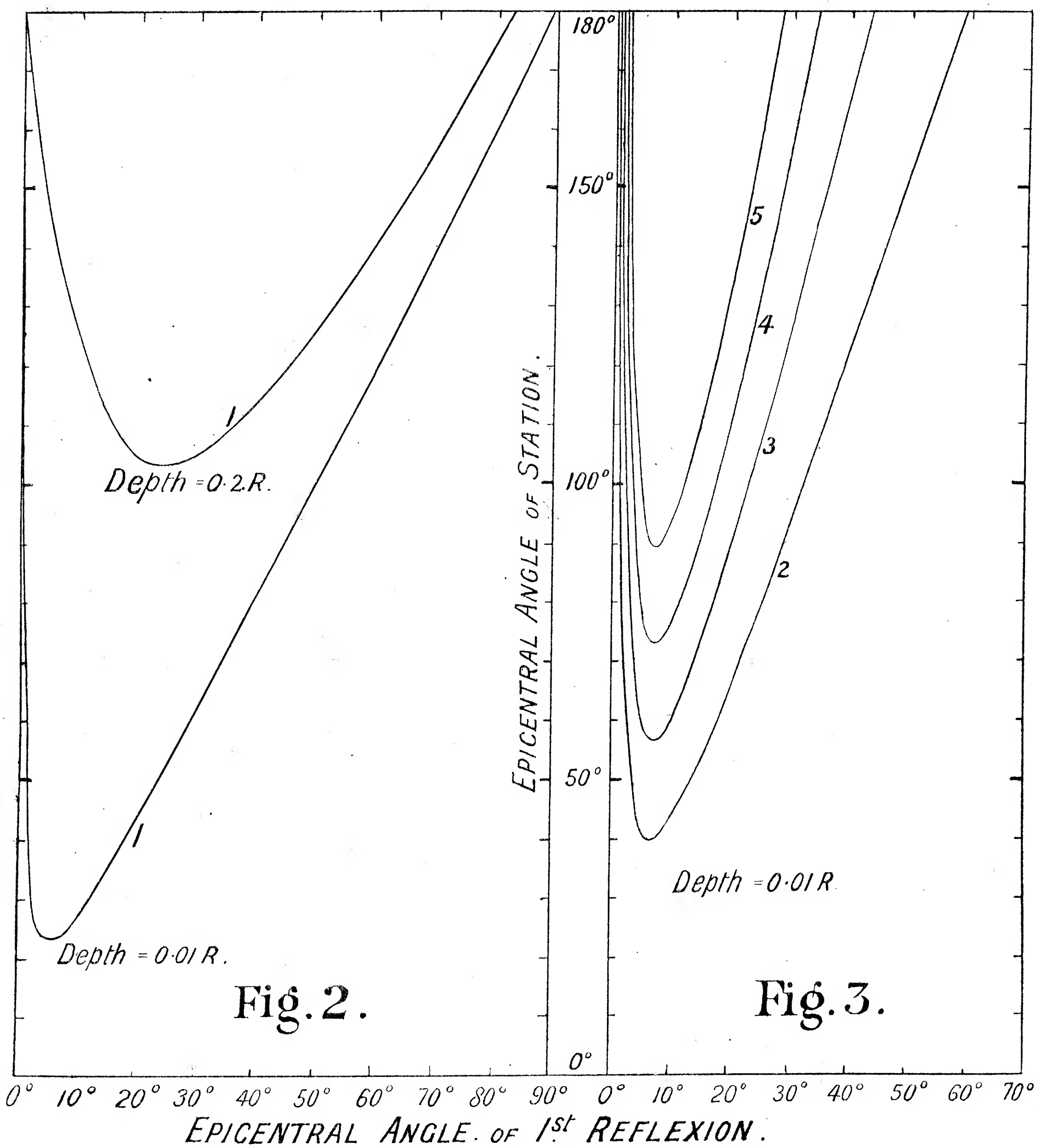


Fig. 2.

PR or SR waves.

Fig. 3.

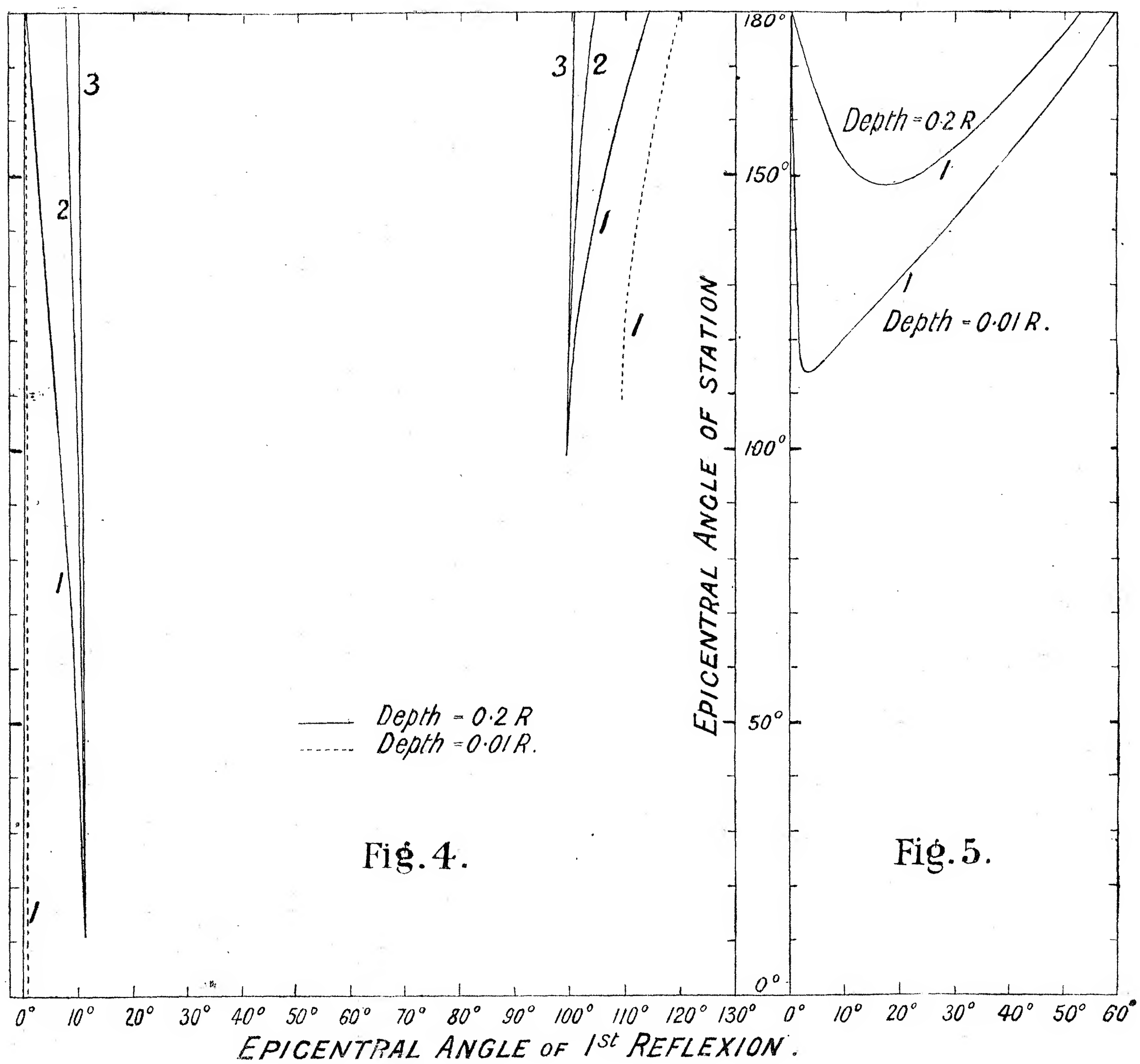
PR_n or SR_n waves.

given Δ. There is no PS_2 until Δ exceeds 180 degrees, and we do not pursue it. The times of PS are included in fig. 1. Fig. 4 explains the position as regards SP . . . SP_∞.

We cannot get an SP wave at all until $\Delta = 11$ degrees, and thereafter it may continue to be shown up to 180 degrees, the point at which reflexion takes place approaching the

epicentre as Δ increases. At $\Delta = 99$ degrees a second SP wave makes its appearance and may continue up to 180 degrees. The times for these are included in fig. 1, dotted lines showing the limiting waves SP_∞.

From $\Delta = 11$ degrees up to $\Delta = 99$ degrees the reflexion of S waves with vibration in the diametral plane through focus and station is complex, and it is thus natural to suppose that this is the region within which the manufacture of Rayleigh waves goes on.



SP_n waves. PS waves.

We should infer that these would not appear until $\Delta = 11$ degrees, but for greater Δ's we should have a continuous succession of contributions of Rayleigh waves starting immediately after S. We need not confuse this with the long-wave phase, which appears to be a crustal phenomenon.

This now completes the effects to be expected for the hypothetical case selected. If it is desired to look into the question of magnitude of the effects at different points,

the investigation might proceed by aid of the tables and diagrams in 'Phil. Trans. Roy. Soc.,' vol. 218, A, p. 373, etc. We shall only point out that, on account of the large angle of impingence in some regions, some of the P effects would only be shown on vertical component seismographs.

The preceding discussion on elementary lines of the effect of finite depth of focus has led to inferences which differ in a very marked degree from those we are accustomed to draw from actual seismograms.

Naturally our results are qualitative in the first instance, and we must be prepared to find quantitative alterations when data for the earth are available. For example, we may be prepared to find that the epicentral distances at which PR, SR or PS start are less than the values we have calculated. In the case of SR or PS this may be so (we have not yet the required data to decide the matter), but in the case of PR, GALITZIN's data, which we have taken as our basis, settle this at once. Anticipating the proof which will be given later, we find that PR starts at 11,000 km., which is only slightly less than the 103 degrees calculated. This, however, implies that the effects hitherto interpreted as PR_1, PR_2, etc., cannot be so described, but in place we may be able to interpret them as SP_1, SP_2, etc. Their capricious occurrence in practice favours this suggestion, and direct test can be made by means of the vertical component seismograph.

There are other possibilities as well as serious difficulties when a deep focus is considered, but we cannot discuss the problem with advantage until we have the requisite data. Hence we now proceed to show what may be deduced from GALITZIN's data themselves and what additional data are wanted before proper tests can be applied.

Fig. 6 shows in graphical form the Pulkovo observed angle of emergence $\bar{e}$ for various

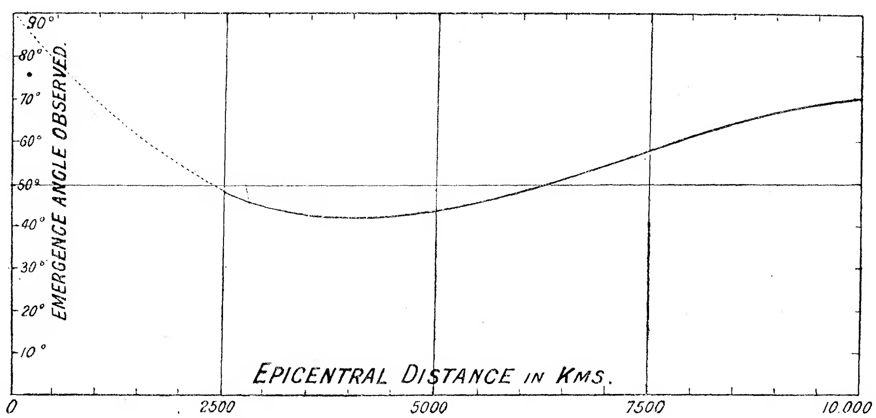


Fig. 6.

epicentral distances Δ. No data are given for $\Delta < 2500$ km., and the dotted part from $\Delta = 0$ to $\Delta = 2500$ km. is hypothetical. It is certain that $\bar{e}$ must be 90 degrees at $\Delta = 0$.

From the formulæ, p. 3, we have

$$V_2 \frac{dT}{d\Delta} = \{\tfrac{1}{2}(1 - \sin \bar{e})\}^{\frac{1}{2}},$$

so that we can at once calculate $V_2 \frac{dT}{d\Delta}$ as a function of Δ from the numbers in fig. 6.

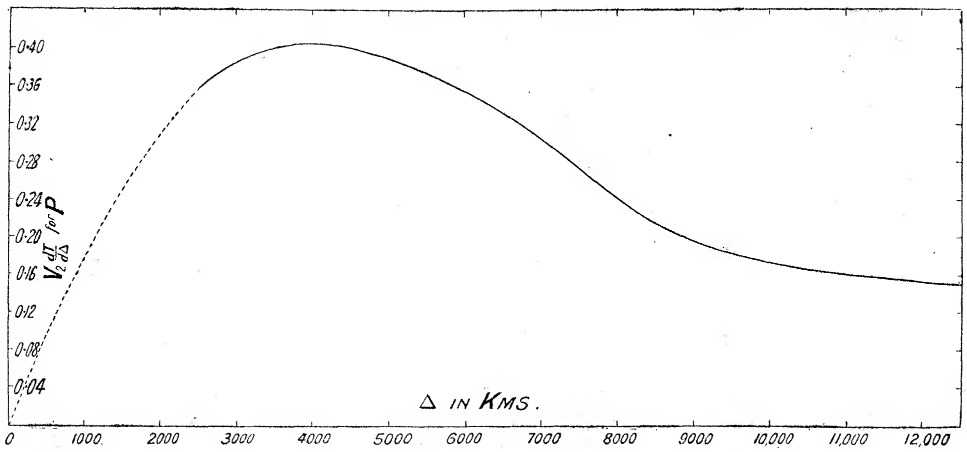


Fig. 7.

The results are shown graphically in fig. 7. Graphical integration of the curve now gives us

$$I = \int_0^{\Delta} \{\tfrac{1}{2}(1 - \sin \bar{e})\}^{\frac{1}{2}} d\Delta$$

as a function of Δ, whence

$$T = A + \frac{I}{V_2}, \text{ where A is a constant,}$$

gives T as a function of Δ.

Table II. gives the value of the integral I for different epicentral distances. Since ZÖPPRITZ's time curve meets with general acceptance for the purpose of determining epicentres, we first seek to see how far we can fit our new time curve with ZÖPPRITZ's. Taking first ZÖPPRITZ's value for V_2, viz., 4·01 km./sec., we find that the two curves fit over the range 6000 km. to 12,000 km. with a discrepancy ± 11 seconds, but the discrepancy rises to 100 seconds at 3000 km. Taking a larger V_2 one can fit the curves together over various ranges. For example, taking $V_2 = 5 \cdot 63$ km./sec. we get the values shown in Table II., where over the range 3500 km. to 8000 km. the discrepancy ranges through only $\pm$ 5 secs., an error we might quite well admit. But large differences must arise towards the epicentre, for on the present view A must be a substantial number representing the time from focus to epicentre. No special significance is to be attached to the above calculation beyond showing that in the middle range of distances we need not make any large departure from ZÖPPRITZ's time curve.

TABLE II.

Δ kms.	$\bar{e}$.	I.	I/5·63 secs.	Z secs.	Z − I/5·63
0	90°	0	0	0	0
500	79	25	4	69	65
1,000	69	95	17	136	119
1,500	61	203	36	199	163
2,000	54	344	61	257	196
2,500	48	512	91	310	219
3,000	44	699	124	358	234
3,500	43	894	159	402	243
4,000	42	1,096	195	442	247
4,500	43	1,299	231	478	247
5,000	44	1,497	266	512	246
5,500	46	1,688	300	542	242
6,000	48	1,871	332	572	240
6,500	51	2,044	363	601	238
7,000	54	2,205	392	631	239
7,500	58	2,351	418	660	242
8,000	62	2,481	441	688	247
8,500	65	2,596	461	716	255
9,000	67	2,700	480	743	263
9,500	68	2,797	497	769	272
10,000	70	2,888	513	795	282
10,500	71	2,972	528	820	292
11,000	72	3,055	543	844	301
11,500	72	3,133	556	867	311
12,000	73	3,209	570	888	318
12,500	73	3,283	583	909	326
13,000	74	3,355	596	929	333

We now proceed to show how a direct test may be applied to the Pulkovo data, and one which will give a determination of A and V_2.

If $\bar{e}$ is the emergence angle of a ray, then the Pulkovo data gives us two distances, say, Δ_1 and Δ_2, for which $\bar{e}$ is the same. We hence infer that a PR wave will be reflected at Δ_1 and pass to epicentral distance $2\Delta_1 + \Delta_2$, and another PR wave will be reflected at Δ_2 and pass to distance $2\Delta_2 + \Delta_1$. *E.g.*, $\bar{e} = 48$ degrees gives $\Delta_1 = 2500$, and $\Delta_2 = 6000$, from which we get distances to station 11,000 km. and 14,500 km.

In this way fig. 8 has been determined directly from the Pulkovo data. It shows that the least distance at which PR occurs is 11,000 km., and for greater distances there are two PR waves for a given epicentral distance. The curve is in very close agreement with the theoretical curve in fig. 2.

The test of the validity of the Pulkovo data is, then, whether for $\Delta > 11{,}000$ km. we can identify the two PR waves on the seismogram. The Pulkovo Bulletins for 1913–1914 show quite a number of records for $\Delta > 11{,}000$ km., and it would seem desirable that a careful study of the seismograms for such distances should be made. Should the search prove successful the curve fig. 8 will then give two distances, Δ_1 and Δ_2, for which

$\Delta = 2\Delta_1 + \Delta_2$ and $\Delta = 2\Delta_2 + \Delta_1$, and there is a check on this by means of the measured $\bar{e}$ for the two waves. Further, if I, I_1 and I_2 are known integrals for Δ, Δ_1 and Δ_2 and T_1 and T_2, the observed time intervals between P and the two PR waves, we have

$$T_1 = 2A + (2I_1 + I_2 - I)/V_2$$

and

$$T_2 = 2A + (2I_2 + I_1 - I)/V_2,$$

which theoretically suffice to determine A and V_2.

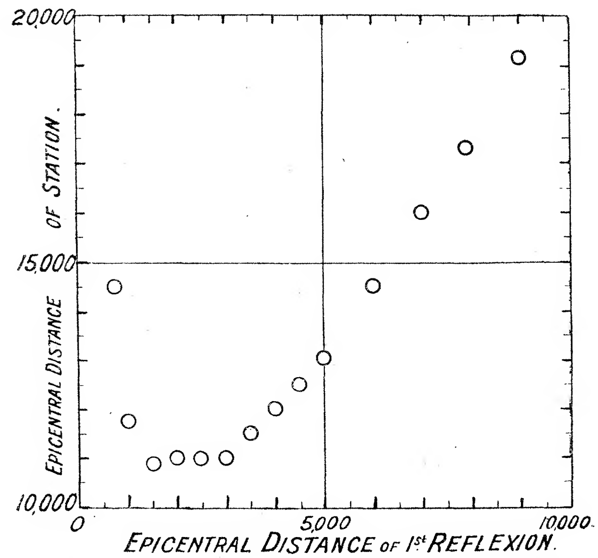


Fig. 8.

It is manifestly a matter of great importance in seismology to settle whether the large depth of focus suggested by the Pulkovo observations can be maintained.

If confirmation is obtained by the research suggested above, the problem of deducing the focal depth and the velocity at any depth must be attacked anew. The two things are related, and it looks as if the process of analysis would be largely tentative. In any case, it is clear that the speed at any depth cannot be uniquely determined until the focal depth is fixed. But one might hazard a guess that a large depth of focus would probably lead to a smaller variation of speed with depth than has been deduced by Zöppritz.

An investigation on S waves might proceed on similar lines, although the relation between the angle of impingence and the apparent angle of emergence is more complex than for P waves; *cf.* 'Phil. Trans.,' *l.c. ante*, p. 378.

The equations are

$$\cos e = V_2 \frac{dT}{d\Delta},$$

$$\tan \bar{e} = \frac{V_2}{V_1} \frac{\sin e' \tan 2e}{\sin e}, \quad \text{where} \quad \cos e' = \frac{V_1}{V_2} \cos e.$$

The matter is further complicated by the circumstance that for a certain range of Δ the reflexion of S waves vibrating in the diametral plane is complex. Some attempts to estimate e for S waves were given in the paper referred to, p. 388, but they are isolated results, and what we require is a systematic investigation for a large range of Δ.

In the absence of such data we may tentatively proceed a little way in the problems arising from finite depth of focus by adding to the time curve obtained from the Pulkovo data the values of S − P given by ZÖPPRITZ, which are known to be not seriously in error over the middle range. Although I have made some calculations in this direction, one cannot proceed very far, and it is an obviously unsatisfactory method.

Summary.

Observations of the emergence angle of P waves at Pulkovo suggest that the depth of focus is of order one-fifth of the earth's radius. It is shown that important modifications would have to be made in the interpretation of seismograms and in the attempt to determine how speed of propagation depends on depth. It is further shown that an important test of the accuracy of the Pulkovo values can be made by a careful scrutiny of seismograms for distances > 11,000 km. Further progress cannot be made until this research has been carried out, and until we have corresponding measures of the angle of emergence of S waves by means of three component seismometers.